under the condition that the constants θ_0, Ω, ω satisfy the relation (2.2) in [9]. Setting up the matrix K, we can show that for the solutions $\theta_0 = 0$, π the system in the first approximation is uncontrollable by the ignorable momenta p_2 and p_3, while for the solution $\theta \neq 0$, π is controllable. The steady-state motion, for which $\theta_0 = 0$, is stable if condition (2.8) in [9] is fulfilled. Such a stable motion can be stabilized up to asymptotic stability by forces of form (22) and minimize an integral of form (25).

BIBLIOGRAPHY

1. Chetaev, N.G., Stability of Motion. Moscow, "Nauka", 1965.
2. Krasovskii, N.N., Theory of Control of Motion. Moscow, "Nauka", 1968.
3. Rumiantsev, V.V., On optimal stabilization of controlled systems. PMM Vol. 34, №3, 1970.
4. Krasovskii, N.N., Problems of the stabilization of controllable motions. In: Malkin, I.G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
5. Lilov, L.K., On the stabilization of steady-state motions of mechanical systems with respect to a part of the variables. PMM Vol. 36, №6, 1972.
6. Lilov, L.K., Determining the smallest number of controls necessary to stabilize the equilibrium position. PMM Vol. 34, №5, 1970.
7. Whittaker, E.T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. New York, Dover Publications, 4th ed., 1944.
8. Gabrielian, M.S. and Krasovskii, N.N., On the problem of stabilization of a mechanical system. PMM Vol. 28, №5, 1964.
9. Rumiantsev, V.V., On the stability of motion of a gyroscope in gimbals. PMM Vol. 22, №3, 1958.

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ON STABILIZATION OF STEADY-STATE MOTIONS OF MECHANICAL SYSTEMS WITH RESPECT TO A PART OF THE VARIABLES

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We pose the problem of stabilization with respect to position coordinates and velocities of the steady-state motions of holonomic mechanical systems by means of forces acting only on the ignorable coordinates. The problem is reduced to the stabilization of the trivial solution of a certain system of differential equations, in which perturbations of the ignorable momenta are treated as the controls. As an example we examine the asymptotic stabilization of the relative equilibrium positions of a gyrostat satellite in a circular orbit.

1. We consider a holonomic scleronomous mechanical system with n degrees of freedom. Let q_r be the generalized coordinates, $q_r^{\cdot}$, p_r $(r = 1, \ldots, n)$ be the generalized velocities and momenta, T and Π be the kinetic and potential energies, respectively, $H = T + \Pi$ be the Hamiltonian function. We assume that besides the potential forces defined by potential Π, nonpotential forces Q_r $(r = 1, \ldots, n)$ also act on the system. We assume that q_α $(\alpha = m + 1, \ldots, n)$ are ignorable coordinates, i.e. $\partial H / \partial q_\alpha = 0$, and that $Q_i \equiv 0$ $(i = 1, \ldots, m)$. Everywhere subsequently the subscripts α and i range over the values indicated above. The Hamiltonian function has the form [1]

$$H = H(q_i, p_i, p_\alpha) = \frac{1}{2} \sum_{r,\, s=1}^{n} c_{rs}(q_1, \ldots, q_m) p_r p_s + \Pi(q_1, \ldots, q_m)$$

Therefore, the system's equations of motion are written as

$$\frac{dq_i}{dt} = \sum_{k=1}^{m} c_{ik}(q) p_k + \sum_{\alpha=m+1}^{m} c_{i\alpha}(q) p_\alpha \tag{1.1}$$

$$\frac{dp_i}{dt} = -\frac{1}{2} \sum_{r,\, s=1}^{n} \frac{\partial c_{rs}(q)}{\partial q_i} p_r p_s - \frac{\partial \Pi(q)}{\partial q_i}, \quad \frac{dp_\alpha}{dt} = Q_\alpha$$

If $Q_\alpha = 0$, the system is found under the action only of the potential forces and can accomplish steady-state motions in which the position coordinates and the ignorable momenta q_i and p_α remain constant, while the ignorable coordinates vary linearly with time.

Suppose that there exists the steady-state motion $q_i = q_i^\circ$, $p_i = p_i^\circ$, $p_\alpha = c_\alpha$. We pose the problem of determining generalized forces Q_α in such a way that this motion would be asymptotically stable relative to a part of the variables q_i and p_i [2]. Without loss of generality we can assume that $q_i^\circ = 0$. The position momenta p_i° are determined from the system of equations (1.1) in which $q_i = 0$, $p_\alpha = c_\alpha$. Let us apply small initial perturbations to the system. Retaining for the values q_i in the perturbed motion the previous notation and letting ξ_i and η_α denote, respectively, the perturbations of the position and the ignorable momenta, $p_i = p_i^\circ + \xi_i$, $p_\alpha = c_\alpha + \eta_\alpha$, after substituting q and p into (1.1) we obtain the equations of perturbed motion

$$\frac{dq_i}{dt} = U_i(q, \xi, \eta) \equiv \sum_{k=1}^{m} c_{ik}(q)(p_k^\circ + \xi_k) + \sum_{\alpha=m+1}^{m} c_{i\alpha}(q)(c_\alpha + \eta_\alpha)$$

$$\frac{d\xi_i}{dt} = V_i(q, \xi, \eta) \equiv -\frac{1}{2} \sum_{j,\, k=1}^{m} \frac{\partial c_{jk}(q)}{\partial q_i} (p_j^\circ + \xi_j)(p_k^\circ + \xi_k) -$$

$$\sum_{j=1}^{m} \sum_{\alpha=m+1}^{n} \frac{\partial c_{j\alpha}(q)}{\partial q_i} (p_i^\circ + \xi_j)(c_\alpha + \eta_\alpha) - \frac{1}{2} \sum_{\alpha,\, \beta=m+1}^{n} \frac{\partial c_{\alpha\beta}(q)}{\partial q_i} \times$$

$$\times (c_\alpha + \eta_\alpha)(c_\beta + \eta_\beta) - \frac{\partial \Pi(q)}{\partial q_i}$$

$$d\eta_\alpha / dt = Q_\alpha \tag{1.2}$$

Thus, the problem posed of the asymptotic stabilization of the steady-state motion $q_i =$ const, $p_i =$ const, $p_\alpha =$ const $(i = 1, \ldots, m;\ \alpha = m + 1, \ldots, n)$ relative to the position coordinates and momenta q_i, p_i with the aid of generalized forces Q_α acting on the ignorable coordinates q_α, is reduced to the problem of the asymptotic stabilization of the trivial solution $q_i = \xi_i = \eta_\alpha = 0$ $(i = 1, \ldots, m;\ \alpha = m + 1, \ldots, n)$ of system (1.2) with $Q_\alpha = 0$ relative to q_i, p_i $(i = 1, \ldots, k)$ with the aid of suitably chosen forces Q_α $(\alpha = m+1, \ldots, n)$.

2. We consider the system

$$dq_i/dt = U_i(q_j, \xi_j, \eta_\alpha), \qquad d\xi_i/dt = V_i(q_j, \xi_j, \eta_\alpha) \tag{2.1}$$

The forces Q_α are not fixed but are subject to determination, therefore, the η_α in (2.1) can be regarded as controls chosen in such a way as to asymptotically stabilize the trivial solution of the system (2.1) of $2m$ equations being considered. If such a choice of $\eta_\alpha = f_\alpha(q_i, \xi_i)$, $f_\alpha(0, 0) = 0$ is possible, the trivial solution of system (2.1) is asymptotically stable under such a choice of η_α. We define forces Q_α by formulas

$$Q_\alpha = \frac{df_\alpha}{dt} = \sum_{i=1}^{m} \left(\frac{\partial f_\alpha}{\partial q_i} U_i + \frac{\partial f_\alpha}{\partial \xi_i} V_i \right) \tag{2.2}$$

The quantity η_α is determined from its own derivative to within an arbitrary constant, therefore,

$$\eta_\alpha = f_\alpha(q_i, \xi_i) + \eta_\alpha^\circ - f_\alpha(q_i^\circ, \xi_i^\circ) \tag{2.3}$$

Here q_i°, ξ_i°, η_α° are the initial perturbations of the position coordinates, the position momenta, and the ignorable momenta, respectively. If we assume that q_i°, ξ_i°, η_α° are sufficiently small (see [3], Sect. 74), the indicated choice (2.2) of forces Q_α ensures the stability of the trivial solution of system (2.1) under condition (2.3) or, what is the same, of the system

$$\frac{dq_i}{dt} = U_i(q_j, \xi_j, f_\alpha) + \sum_{\alpha=m+1}^{n} \frac{\partial U_i(q_j, \xi_j, f_\alpha)}{\partial \eta_\alpha} (\eta_\alpha^\circ - f_\alpha^\circ)$$

$$\frac{d\xi_i}{dt} = V_i(q_j, \xi_j, f_\alpha) + \sum_{\alpha=m+1}^{n} \frac{\partial V_i(q_j, \xi_j, f_\alpha)}{\partial \eta_\alpha} (\eta_\alpha^\circ - f_\alpha^\circ) +$$

$$\frac{1}{2} \sum_{\alpha, \beta=m+1}^{n} \frac{\partial^2 V_i(q_j, \xi_j, f_\alpha)}{\partial \eta_\alpha \partial \eta_\beta} (\eta_\alpha^\circ - f_\alpha^\circ)(\eta_\beta^\circ - f_\beta^\circ) \tag{2.4}$$

$$(f_\alpha^\circ = f_\alpha(q_i^\circ, \xi_i^\circ))$$

Indeed, according to the assumption made, system (2.1) is asymptotically stable, while system (2.4) differs from system (2.1) by the presence of constantly acting perturbations which can be made arbitrarily small along with q_i°, ξ_i°, η_α°.

In order to achieve the asymptotic stability of the trivial solution of system (2.1) we assume that the forces Q_α are impulsive [4]. By introducing the Dirac δ-function and defining the forces acting on the ignorable coordinates by the formulas

$$Q_\alpha = \frac{df_\alpha}{dt} + \delta(t - t_0)(f_\alpha^\circ - \eta_\alpha^\circ)$$

we obtain the required values of $\eta_\alpha = f_\alpha(q_i, \xi_i)$ for the perturbations of the ignorable momenta.

Let us consider the first approximation of the equations of perturbed motion

$$dq/dt = L_1 q + L_2 \xi + B_1 \eta$$
$$d\xi/dt = L_3 q - L_1^* \xi + B_2 \eta \tag{2.5}$$

Here

$$q^* = \| q_1, q_2, \ldots, q_m \|, \qquad \xi^* = \| \xi_1, \xi_2, \ldots, \xi_m \|$$
$$\eta^* = \| \eta_{m+1}, \eta_{m+2}, \ldots, \eta_n \|$$

$$L_1 = \left\| \frac{\partial^2 H}{\partial p_i \partial q_j} \right\|_{i,\, j=1}^{m}, \quad L_2 = \left\| \frac{\partial^2 H}{\partial p_i \partial p_j} \right\|_{i,\, j=1}$$

$$L_3 = -\left\| \frac{\partial^2 H}{\partial q_i \partial q_j} \right\|_{i,\, j=1}^{m}, \quad B_1 = \left\| \frac{\partial^2 H}{\partial p_i \partial p_\alpha} \right\|_{i=1,\, \alpha=m+1}^{m \quad n}$$

$$B_2 = -\left\| \frac{\partial^2 H}{\partial q_i \partial p_\alpha} \right\|_{i=1,\, \alpha=m+1}^{m \quad n}, \quad L = \left\| \begin{matrix} L_1 & L_2 \\ L_3 & -L_1^* \end{matrix} \right\|, \quad B^* = \| B_1, B_2 \|$$

The values of the derivatives of the functions are taken at the point $q_i = 0, p_i = p_i^\circ$, $p_\alpha = c_\alpha$; here and subsequently the asterisk denotes transposition. The matrices L_2 and L_3 are symmetric, therefore, the characteristic equation of system (2.5),

$$\det \left\| \begin{matrix} L_1 - \lambda E & L_2 \\ L_3 & -L_1^* - \lambda E \end{matrix} \right\| = 0 \tag{2.6}$$

where E is the unit $m \times m$ matrix, does not change when λ is replaced by $-\lambda$ and, consequently, contains only even powers of λ. This signifies that the greatest common divisors of the ith-order minors $D_i\ (\lambda)$ of the characteristic matrix of system (2.5), which are not identically equal to unity, have roots with nonnegative real parts. Therefore, to achieve asymptotic stability of the trivial solution of system (2.5) by a certain control η, it is necessary [5] that

$$\text{rank}\ \| B, LB, \ldots, L^{2m-1} B \| = 2m \tag{2.7}$$

But condition (2.7) is a necessary and sufficient condition for the complete controllability of system (2.5) (see [4]). Thus, the question of the asymptotic stabilization of the trivial solution of system (2.5) coincides with the question of the complete controllability of system (2.5). Consequently, if the solution $q = \xi = 0$ is asymptotically stabilizable, the system can be led to the origin in a finite time interval and in such a way that a certain preassigned functional is minimized on this motion [4]. For the complete system (2.1) condition (2.7) can be necessary only if among the roots of Eq. (2.6) there are roots with positive real parts [5]. Generally speaking, the addition of higher-order terms can strengthen the stability in case rank $\| B, LB, \ldots, L^{2m-1}B \| < 2m$, if it exists, up to asymptotic stability. Thus, we can state the following proposition.

Theorem. In order that a certain steady-state motion $q_i = \text{const}\ (i = 1, \ldots, m)$ can be asymptotically stabilized relative to the position coordinates and position momenta, q_i, p_i, by means of forces acting on the ignorable coordinates q_α $(\alpha = m + 1, \ldots, n)$, it is sufficient that the rank of the matrix $\| B, LB, \ldots, L^{2m-1}B \|$, where matrices B and L have the form (2.5), be equal to $2m$. This condition can be necessary only if among the roots of Eq. (2.6) there are roots with positive real parts.

3. As an example of the proposed method for the asymptotic stabilization of steady-state motions of mechanical systems we consider the problem of the asymptotic stabilization of the relative equilibrium positions of a gyrostat satellite by means of flywheels. This problem is of independent interest. We assume that the center of gravity of the gyrostat satellite describes a circular orbit in a Newtonian force field. We examine the restricted problem, neglecting the influence of the motion around the center of mass on the motion of the center of mass. As the origin of an inertial coordinate system $O_1\xi\eta\zeta$ we take the center of attraction O_1, and as the origin of a moving coordinate system

$Ox_1x_2x_3$ we take the center of mass O of the satellite and we direct the axes along the principal central axes of inertia. We introduce one more moving coordinate system $Oxyz$, whose z-axis is directed along the straight line O_1O, the x-axis is directed to the side of motion of the center of mass along a staight line orthogonal to the z-axis and located in the orbital plane, the y-axis completes the x- and z-axes to a right trihedron. The position of the satellite's body in the orbital coordinate system $Oxyz$ is determined by the coordinates q_i $(i = 1, 2, 3)$, as which we take the Euler angles ψ, θ, φ. The cosines of the angles between the systems $Oxyz$ and $Ox_1x_2x_3$ are given by

$$\cos(x, x_i) = \alpha_i, \quad \cos(y, x_i) = \beta_i, \quad \cos(z, x_i) = \gamma_i$$
$$\alpha_1 = \cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta$$
$$\alpha_2 = -\sin\varphi\cos\psi - \cos\varphi\sin\psi\cos\theta, \quad \alpha_3 = \sin\theta\sin\psi$$
$$\beta_1 = \cos\varphi\sin\psi + \sin\varphi\cos\psi\cos\theta$$
$$\beta_2 = -\sin\varphi\sin\psi + \cos\varphi\cos\psi\cos\theta, \ \beta_3 = -\sin\theta\cos\psi$$
$$\gamma_1 = \sin\varphi\sin\theta, \quad \gamma_2 = \cos\varphi\sin\theta, \quad \gamma_3 = \cos\theta$$

For simplicity of computation, in what follows we assume that the the gyrostat has three rotors directed along the principal axes of inertia. The angles of rotation of the rotors relative to the satellite's body are denoted δ_s $(s = 1, 2, 3)$. The equations of motion of the gyrostat satellite in the $Oxyz$ system, under the assumption that its center of mass moves in a circular orbit, can be written in the form of Hamiltonian equations, where

$$q_1 = \psi, \ q_2 = \theta, \ q_3 = \varphi, \ q_4 = \delta_1, \ q_5 = \delta_2, \ q_6 = \delta_3 \quad (n = 6)$$

$$H = -\omega_0 \frac{\cos\psi\cos\theta}{\sin\theta} p_1 - \omega_0 \sin\psi\, p_2 + \omega_0 \frac{\cos\psi}{\sin\theta} p_3 + $$
$$+ \frac{1}{2} p^* A p + \frac{3}{2} \omega_0 \sum_{s=1}^{3} A_s \gamma_s^2$$

Here ω_0 is the angular velocity of revolution of the satellite along the orbit, A_s is the sth principal moment of inertia of the satellite. The matrix A has the following elements:

$$A = \| a_{ij} \|, \quad a_{ij} = a_{ji} \quad (i, j = 1, \ldots, 6)$$
$$a_{11} = \frac{h_1\cos^2\varphi + h_2\sin^2\varphi}{h_1h_2\sin^2\theta}, \quad a_{12} = \frac{h_2 - h_1}{h_1h_2\sin\theta}\sin\varphi\cos\varphi$$
$$a_{13} = -\frac{\cos\theta}{h_1h_2\sin^2\theta}(h_1\cos^2\varphi + h_2\sin^2\varphi), \quad a_{14} = -\frac{\sin\varphi}{h_1\sin\theta}$$
$$a_{15} = -\frac{\cos\varphi}{h_2\sin\theta}, \quad a_{22} = \frac{h_1\sin^2\varphi + h_2\cos^2\varphi}{h_1h_2}$$
$$a_{23} = \frac{h_1 - h_2}{h_1h_2\sin\theta}\sin\varphi\cos\varphi, \quad a_{24} = -\frac{\cos\varphi}{h_1}, \quad a_{25} = \frac{\sin\varphi}{h_2}$$
$$a_{33} = \frac{1}{h_3} + \frac{\cos^2\theta}{h_1h_2\sin^2\theta}(h_1\cos^2\varphi + h_2\sin^2\varphi), \ a_{34} = \frac{\sin\varphi\cos\theta}{h_1\sin\theta}$$
$$a_{35} = \frac{\cos\varphi\cos\theta}{h_2\sin\theta}, \ a_{36} = -\frac{1}{h_3}, \quad a_{44} = \frac{A_1}{I_1h_1}, \quad a_{55} = \frac{A_2}{I_2h_2}$$
$$a_{66} = \frac{A_3}{I_3h_3}, \ a_{16} = a_{26} = a_{45} = a_{66} = 0$$
$$h_s = A_s - I_s \quad (s = 1, 2, 3)$$

In these formulas I_s is the moment of inertia of the sth rotor. The generalized nongravitational forces Q_i $(i = 1, 2, 3)$ are taken as zero in what follows. We see that the coordinates δ_s $(s = 1, 2, 3)$ do not occur in the expression for H, i.e. p_{s+3} are ignorable momenta. Therefore, to study the relative motions of the satellite we can make use of Eqs. (1.1) in which now $m = 3$. The set of relative equilibrium positions was completely determined in [6]. We assume that $A_1 \neq A_2 \neq A_3$. We can show [7] that this set is defined by the equation

$$A_1\alpha_1\gamma_1 + A_2\alpha_2\gamma_2 + A_3\alpha_3\gamma_3 = 0 \tag{3.1}$$

and that all the relative equilibrium positions of the gyrostat satellite fall into three classes [8].

3.1. One of the satellite's principal axes of inertia, say A_2, is collinear with the axis Oz,

$$\theta = (2k + 1)^{1/2}\pi \ (k = 0,1), \quad \varphi = s\pi \ (s = 0, 1), \quad 0 \leqslant \psi \leqslant 2\pi$$

3.2. One of the satellite's principal axes of inertia, say A_1, is collinear with the axis Ox,

$$\psi = k\pi \ (k = 0, 1), \quad \varphi = s\pi \ (s = 0, 1), \quad 0 < \theta < \pi$$

3.3. None of the satellite's principal axes of inertia is collinear with the axes of the orbital coordinate system,

$$\operatorname{ctg}\psi = \frac{M + \sin^2\varphi}{\sin\varphi\cos\varphi}\cos\theta, \quad M = \frac{A_2 - A_3}{A_1 - A_2}, \quad \prod_{i=1}^{3}\alpha_i\gamma_i \neq 0$$

In the case being considered the reduced potential energy (the Routh potential) W [1, 9] has the form [9]

$$-W = \frac{\omega_0^2}{2}\sum_{s=1}^{3} h_s\beta_s^2 - \frac{3}{2}\omega_0^2\sum_{s=1}^{3} A_s\gamma_s^2 + \omega_0\sum_{s=1}^{3} p_{s+3}\beta_s - \frac{1}{2}\sum_{s=1}^{3}\frac{p_{s+3}^2}{I_s}$$

The quantities $\psi, \theta, \varphi, c_{s+3}$ $(s = 1, 2, 3)$ must satisfy the equations

$$-\frac{\partial W}{\partial\psi} = \omega_0^2\sum_{s=1}^{3} h_s\beta_s\alpha_s + \omega_0\sum_{s=1}^{3} c_{s+3}\alpha_s = 0$$

$$-\frac{\partial W}{\partial\theta} = -\omega_0^2\cos\psi\sum_{s=1}^{3} h_s\beta_s\gamma_s - 3\omega_0^2\sin\theta\cos\theta(A_1\cos^2\varphi + A_2\cos^2\varphi - A_3) -$$

$$-\omega_0\cos\psi\sum_{s=1}^{3} c_{s+3}\gamma_s = 0$$

$$-\frac{\partial W}{\partial\varphi} = \omega_0^2\beta_1\beta_2(h_1 - h_2) - 3\omega_0^2(A_1 - A_2)\gamma_1\gamma_2 + \omega_0(c_4\beta_2 - c_5\beta_1) = 0$$

From these equations, taking 3.1 into account, we obtain the following expressions for c_{s+3}:

$$\omega_0^{-1}c_{s+3} = (\chi - h_s)\beta_s - a\gamma_s, \quad a = 3\sum_{s=1}^{3} A_s\gamma_s\beta_s$$
$$(s = 1, 2, 3)$$

In these formulas χ is an arbitrary parameter and, consequently, the constant values of the ignorable momenta c_{s+3} $(s = 1, 2, 3)$ are determined nonuniquely at any relative equilibrium position.

In the problem being examined the matrices L_1, L_2, L_3, B have, as can be shown, the following elements:

$$L_1 = \| l_{ik}^1 \|, \quad L_2 = \| l_{ik}^2 \| \quad (l_{ik}^2 = l_{ik}^2)$$

$$L_3 = \| l_{ik}^3 \| \quad (l_{ik}^3 = l_{ki}^3), \quad B = \| b_1, b_2, b_3 \| = \| b_{sk} \|$$

$$(i, k = 1, 2, 3; \quad s = 1, \ldots, 6)$$

$$l_{11}{}^1 = \omega_0 \sin\psi \quad \omega \operatorname{tg}\theta, \quad l_{12}{}^1 = \omega_0, \quad \left[\cos\psi - \chi \frac{\cos\psi}{\sin^2\theta}\left(\frac{\sin^2\varphi}{h_1} + \frac{\cos^2\varphi}{h_2}\right)\right]$$

$$l_{21}{}^1 = -\omega_0 \cos\psi, \quad l_{22}{}^1 = \omega_0 \frac{h_2 - h_1}{h_1 h_2 \sin^2\theta} \sin\varphi\cos\varphi\,(a\cos\theta + \chi\sin\psi)$$

$$l_{31}{}^1 = -\omega_0 \frac{\sin\psi}{\sin\theta}, \quad l_{32}{}^1 = \omega_0 \frac{\cos\theta\cos\psi}{\sin^2\theta}\chi\left(\frac{\sin^2\varphi}{h_1} + \frac{\cos^2\varphi}{h_2}\right)$$

$$l_{13}{}^1 = \omega_0\left[\frac{\sin\psi}{\sin\theta} + \frac{\chi}{\sin\theta}\left(\frac{\beta_2\sin\varphi}{h_1} - \frac{\beta_1\cos\varphi}{h_2}\right) + a\sin\varphi\cos\varphi\left(\frac{1}{h_2} - \frac{1}{h_1}\right)\right]$$

$$l_{23}{}^1 = \omega_0\left[-\cos\psi\cos\theta + \chi\left(\frac{\beta_2\cos\varphi}{h_1} + \beta_1\frac{\sin\varphi}{h_2}\right) - a\sin\theta\left(\frac{\cos^2\varphi}{h_1} + \frac{\sin^2\varphi}{h_2}\right)\right]$$

$$l_{33}{}^1 = \omega_0\left[-\frac{\cos\theta\sin\psi}{\sin\theta} + \chi\operatorname{ctg}\theta\left(\frac{\beta_1\cos\varphi}{h_2} - \frac{\beta_2\sin\varphi}{h_1}\right) + \right.$$
$$\left. + a\cos\theta\sin\varphi\cos\varphi\left(\frac{1}{h_1} - \frac{1}{h_2}\right)\right]$$

$$l_{11}{}^2 = \frac{h_1\cos^2\varphi + h_2\sin^2\varphi}{h_1 h_2\sin^2\theta}, \qquad l_{12}{}^2 = \frac{h_2 - h_1}{h_1 h_2 \sin\theta}\sin\varphi\cos\varphi$$

$$l_{13}{}^2 = -\frac{\cos\theta}{h_1 h_2\sin^2\theta}(h_1\cos^2\varphi + h_2\sin^2\varphi)$$

$$l_{22}{}^2 = \frac{h_1\sin^2\varphi + h_2\cos^2\varphi}{h_1 h_2}$$

$$l_{23}{}^2 = \frac{h_1 - h_2}{h_1 h_2 \sin\theta}\sin\varphi\cos\varphi\cos\theta$$

$$l_{33}{}^2 = \frac{1}{h_3} + \frac{\cos^2\theta}{h_1 h_2 \sin^2\theta}(h_1\cos^2\varphi + h_2\sin^2\varphi)$$

$$l_{11}{}^3 = -\omega_0{}^2\chi, \quad l_{12}{}^3 = \omega_0{}^2(\sin\psi\cos\psi\operatorname{ctg}\theta\chi - a\sin\psi), \quad l_{13}{}^3 = 0$$

$$l_{22}{}^3 = -\omega_0{}^2\left[\frac{\cos^2\psi}{\sin^2\theta}\left(\frac{\sin^2\varphi}{h_1} + \frac{\cos^2\varphi}{h_2}\right)\chi^2 - \cos^2\psi\chi + \right.$$
$$\left. 3\cos 2\theta\,(A_1\sin^2\varphi + A_2\cos^2\varphi - A_3)\right]$$

$$l_{23}{}^3 = -\omega_0{}^2\left\{\frac{\cos\psi}{\sin\theta}\left(\frac{\cos\varphi\,\beta_1}{h_2} - \frac{\sin\varphi\,\beta_2}{h_1}\right)\chi^2 + \right.$$
$$\left[-\frac{\sin\psi\cos\psi}{\sin\theta} + a\sin\varphi\cos\varphi\cos\psi\left(\frac{1}{h_1} - \frac{1}{h_2}\right)\right]\chi +$$
$$\left. 6\sin\varphi\cos\varphi\sin\theta\cos\theta\,(A_1 - A_2)\right\}$$

$$l_{33}{}^3 = -\omega_0{}^2\left\{\left(\frac{\beta_2{}^2}{h_1} + \frac{\beta_1{}^2}{h_2}\right)\chi^2 - \left[1 - \beta_3{}^2 + 2a\left(\frac{\beta_2\gamma_2}{h_1} + \frac{\beta_1\gamma_1}{h_2}\right)\right]\chi - \right.$$
$$\left. a\beta_3\gamma_3 + a^2\left(\frac{\gamma_2{}^2}{h_1} + \frac{\gamma_1{}^2}{h_2}\right) + 3\sin^2\theta\cos 2\varphi\,(A_1 - A_2)\right\}$$

$$b_{11} = -\frac{\sin\varphi}{h_1\sin\theta}, \quad b_{12} = -\frac{\cos\varphi}{h_2\sin\theta}, \quad b_{21} = -\frac{\cos\varphi}{h_1}$$

$$b_{22} = \frac{\sin\varphi}{h_2}, \quad b_{31} = \frac{\sin\varphi\cos\theta}{h_1\sin\theta}, \quad b_{32} = \frac{\cos\varphi\cos\theta}{h_2\sin\theta}$$

$$b_{33} = -\frac{1}{h_3}, \quad b_{51} = -\frac{\omega_0 \sin\varphi \cos\psi}{h_1 \sin\theta}\chi, \quad b_{52} = -\frac{\omega_0 \cos\varphi \cos\psi}{h_2 \sin\theta}\chi$$

$$b_{61} = -\frac{\omega a}{h_1}\gamma_2 + \frac{\omega_0\beta_2}{h_1}\chi, \qquad b_{62} = \frac{\omega_0 a}{h_2}\gamma_1 - \frac{\omega_0\beta_1}{h_2}\chi$$

$$b_{13} = b_{23} = b_{41} = b_{42} = b_{43} = b_{53} = b_{63} = 0$$

If a certain relative equilibrium position of the satellite proves to be stabilizable, then, according to Sect. 2, this signifies that by rotating the rotors in a suitable manner we can achieve the asymptotic stability of the relative equilibrium position being considered. In other words, any sufficiently small perturbations of the relative equilibrium point being stabilized can be "damped" by moments applied to the flywheels and the system led to an equilibrium state in a finite time interval. The possibility of asymptotic stabilization of the given stationary point is determined by the rank of the matrix

$$C = \| B, LB, \ldots, L^5 B \|$$

Let us investigate the rank of matrix C on families 1 and 2. For points of family 1 we have

$$\det \| b_1,\ b_2\ b_3,\ Lb_1,\ L^2 b_1,\ Lb_3 \| =$$

$$\pm 27 \frac{\omega_0^7}{h_1^3 h_2 h_3^2} \cos\psi (A_1 - A_2)(A_2 - A_3)^2$$

$$\det \| b_1,\ b_2\ b_3,\ Lb_1,\ L^2 b_3,\ Lb_3 \| =$$

$$\pm 27 \frac{\omega_0^7}{h_1^2 h_2 h_3^3} \sin\psi (A_1 - A_2)^2 (A_2 - A_3)$$

We have either $\sin\psi \neq 0$ or $\cos\psi \neq 0$, therefore, all stationary points of family 1, which can be represented geometrically as a rotation through an arbitrary angle around one of the satellite's principal axes of inertia, collinear with axis $\dot{O}z$, can be made asymptotically stable by moments applied to the flywheels. At points of family 2

$$\det \| b_1,\ b_2,\ b_3,\ Lb_1,\ L^2 b_1,\ Lb_3 \| =$$

$$\pm \frac{27\omega_0^7}{h_1^3 h_2 h_3^2} \sin\theta \cos^2 2\theta (A_1 - A_2)(A_2 - A_3)^2$$

Thus, all relative equilibrium positions of family 2, which are obtained one from the other by a rotation through an angle θ around the axis Ox_1, collinear with the axis Ox, can be asymptotically stabilized by moments applied to the flywheels, except for the cases $\theta = \pi / 4,\ 3\pi / 4$. We can show that in these cases the rank of matrix C equals four and, consequently, the points $\theta = \pi / 4,\ 3\pi / 4$ are uncontrollable. The possibility of stabilizing them is determined by terms of higher than the first order of smallness, since when $\cos 2\theta = 0$ the matrix L has, as can be shown after cumbersome calculations, five eigenvalues with nonnegative real parts and the spectrum of the 4×4 matrix Q, indicated in [10], cannot contain all of them.

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BIBLIOGRAPHY

1. Gantmakher, F. R., Lectures on Analytical Mechanics. Moscow, "Nauka", 1966.
2. Rumiantsev, V. V., On stability of motion relative to a part of the variables. Vestnik Moskovsk. Gos. Univ., Ser. Mat. Mekhan., Astronom. Fiz., Khim, №4, 1957.

3. Malkin, I. G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
4. Krasovskii, N. N., Theory of Stability of Motion. Moscow, "Nauka", 1968.
5. Lilov, L. K., Determining the smallest number of controls necessary to stabilize the equilibrium position. PMM Vol. 34, №5, 1970.
6. Stepanov, S. Ia., On the steady motions of a gyrostat satellite. PMM Vol. 33, №1, 1969.
7. Anchev, A. A., On the stability of relative equilibrium of a satellite with rotors. B"lgarska Akademiia na Naukite, Isvestiia na Matematicheskii Institut, Vol. 11, 1969.
8. Morozov, V. M., On the stability of relative equilibrium of a satellite under the action of graviational, magnetic, and aerodynamical moments. Kosmicheskie Issledovaniia, Vol. 7, №3, 1969.
9. Rumiantsev, V. V., On the stability of Steady-State Motions of Satellites. Vychisl. Tsentr Akad. Nauk SSSR, 1967.
10. Gal'perin, E. A. and Krasovskii, N. N., On the stabilization of stationary motions in nonlinear control systems. PMM Vol. 27, №6, 1963.

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EXTREMAL CONTROL IN A NONLINEAR DIFFERENTIAL GAME

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We consider the game problem of the encounter of a conflict-controlled phase point with a given set. We prove sufficient conditions for the successful completion of a nonlinear game of encounter. These conditions are based on the idea of minimax extremal aiming [1]. The given aiming is realized here on the basis of absorption sets [2]. These sets are constructed with the aid of auxiliary motions generated by program controls which are represented by suitable Borel measures in accordance with the well known techniques [3] of generalized solutions of ordinary differential equations.

1. Statement of the problem. We consider a controlled system described by the vector differential equation

$$\dot{x} = f(t, x, u, v) \tag{1.1}$$

Here x is the system's n-dimensional phase vector; u and v are r-dimensional vector controls of the first and second players, respectively, constrained by the conditions $u \in P$, $v \in Q$, where P and Q are bounded closed sets. The function $f(t, x, u, v)$ is assumed continuous for all argument values to be considered and satisfies a Lipschitz condition in x in every bounded region of the space $\{x\}$. Furthermore, the following conditions for the continuability of the solutions $x[t]$ for Eq. (1.1) are assumed to be fulfilled. Let $F(t, x) = \text{co}^* \{f(t, x, u, v): u \in P, v \in Q\}$, where $\text{co}^* \{f\}$